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TOPOLOGICAL PROPERTIES OF PARANORMAL OPERATORS ON HILBERT SPACE

BY

GLENN R. LUECKE

ABSTRACT. Let $B(H)$ be the set of all bounded endomorphisms (operators) on the complex Hilbert space H. $T \in B(H)$ is paranormal if $\|(T - zI)^{-1}\| = 1/d(z, \sigma(T))$ for all $z \notin \sigma(T)$ where $d(z, \sigma(T))$ is the distance from z to $\sigma(T)$, the spectrum of T. If $\mathcal{P}$ is the set of all paranormal operators on H, then $\mathcal{P}$ contains the normal operators, $\mathfrak{N}$, and the hyponormal operators; and $\mathcal{P}$ is contained in $\mathfrak{L}$, the set of all $T \in B(H)$ such that the convex hull of $\sigma(T)$ equals the closure of the numerical range of T. Thus, $\mathfrak{N} \subseteq \mathcal{P} \subseteq \mathfrak{L} \subseteq B(H)$. Give $B(H)$ the norm topology. The main results in this paper are (1) $\mathfrak{N}$, $\mathcal{P}$, and $\mathfrak{L}$ are nowhere dense subsets of $B(H)$ when $\dim H \geq 2$, (2) $\mathfrak{N}$, $\mathcal{P}$, and $\mathfrak{L}$ are arcwise connected and closed, and (3) $\mathfrak{N}$ is a nowhere dense subset of $\mathcal{P}$ when $\dim H = \infty$.

Paranormal operators have received considerable attention in the current literature ([16], [17], [19], [20], [21], [23], [24], [25]). However, only the various spectral properties of paranormal operators have been discussed. In this paper, the topological properties of the set of all paranormal operators $\mathcal{P}$ on a Hilbert space H are investigated.

We begin by giving the notation to be used and by defining some of the more specialized terminology. The point spectrum and approximate point spectrum of an operator T are denoted by $\sigma_p(T)$ and $\sigma_\pi(T)$, respectively. $z \in \sigma_p(T)$ is a normal eigenvalue if $\{x \in H: (T - zI)x = 0\} = \{x \in H: (T - zI)^*x = 0\}$ where I denotes the identity operator on H. $z \in \sigma_\pi(T)$ is a normal approximate eigenvalue of T when (1) $\|x_n\| = 1$ and $\|(T - zI)x_n\| \to 0$ as $n \to \infty$ imply $\|(T - zI)^* x_n\| \to 0$ as $n \to \infty$, and (2) $\|y_n\| = 1$ and $\|(T - zI)^* y_n\| \to 0$ as $n \to \infty$ imply $\|(T - zI)y_n\| \to 0$ as $n \to \infty$. If S is a set of complex numbers, then ∂S denotes the boundary of S and $\mathrm{co}(S)$ denotes the convex hull of S. Let $W(T) = \text{closure}\{(Tx, x): x \in H, \|x\| = 1\}$ denote the closure of the numerical range of T. Let $R(T, z) = (T - zI)^{-1}$ for each $z \in \rho(T)$, the resolvent set of T. The spectral radius of T is denoted by $R_{\mathrm{sp}}(T)$. $T \in B(H)$ is hyponormal if $T^*T - TT^* \geq 0$.

I. Elementary properties of paranormal operators.

Stampfli [19] has shown that every hyponormal operator is paranormal. Therefore, since every normal

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operator is hyponormal, $\mathfrak{N} \subseteq \mathcal{P}$. G. Orland [13] proved that $T \in \mathfrak{L}$ if and only if $\|R(T, z)\| \leq 1/d(z, \text{co}\,\sigma(T))$ for all $z \notin \text{co}\,\sigma(T)$. From this it follows immediately that if T is a paranormal operator, then $\text{co}\,\sigma(T) = W(T)$, i.e. $\mathcal{P} \subseteq \mathfrak{L}$.

Theorem 1.1. *If T is a paranormal operator and if α and β are complex numbers, then $\alpha T + \beta I$ and T^* are paranormal.*

The proof is a simple computation that depends on $\sigma(\alpha T + \beta I) = \alpha\sigma(T) + \beta$ and $\sigma(T^*) = \sigma(T)^*$.

The following theorem is very useful in constructing examples.

Theorem 1.2. *If A is any operator on H, then there exists a Hilbert space K and a normal operator N on K such that $T = A \oplus N \in B(H \oplus K)$ is paranormal.*

Proof. Since $W(A)$ is a compact set of complex numbers, there exists a Hilbert space K and a normal operator N on K such that $\sigma(N) = W(A)$ [3, p. 581]. Let $T = A \oplus N$. Then $\sigma(T) = \sigma(N) = W(A)$. For $z \notin \sigma(T) = W(A)$, it is well known that $\|R(A, z)\| \leq 1/d(z, W(A)) = 1/d(z, \sigma(T))$. Since N is normal and $z \in \rho(N)$, $\|R(N, z)\| = 1/d(z, \sigma(T))$. Therefore

$$\|R(T, z)\| = \text{Max}\{\|R(A, z)\|, \|R(N, z)\|\} = 1/d(z, \sigma(T)).$$

Thus T is paranormal.

As we shall see, the class of all hyponormal operators on H is distinct, in general, from the class $\mathcal{P}$ of paranormal operators. We know [19] that if T is hyponormal, then $\|T\| = R_{sp}(T)$ and if T is also invertible, then T^{-1} is hyponormal. These properties do not generalize to paranormal operators.

Theorem 1.3. *There exists an invertible paranormal operator T such that*

1. *T is not hyponormal,*
2. *T^2 is not paranormal,*
3. *$\|T\| > R_{sp}(T)$, and*
4. *T^{-1} is not paranormal.*

Proof. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Let N be a normal operator such that $\sigma(N) = W(A)$, and let $T = A \oplus N$. Then by Theorem 1.2, T is paranormal. We know from [19] that a hyponormal operator is hyponormal on invariant subspaces. Therefore, since A is not hyponormal, T is not hyponormal. By [1], $W(A)$ is the closed disc of radius $1/2$ about $z = 1$, and $W(A^2)$ is the closed disc of radius 1 about $z = 1$. Therefore

$$0 \in W(A^2) \subseteq W(T^2) \quad \text{and} \quad 0 \notin \text{co}(\sigma(T)^2) = \text{co}\,\sigma(T^2).$$

Therefore, $\text{co}\,\sigma(T^2) \neq W(T^2)$ and so T^2 is not paranormal. Let $x = \begin{bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{bmatrix}$, then $\|x\| = 1$ and $\|Ax\| = \sqrt{10}/2$. Then

$$\|T\| \geq \|Ax\| = \sqrt{10}/2 > 3/2 = R_{sp}(T).$$

Therefore $\|T\| > R_{sp}(T)$. If T^{-1} were paranormal, then

$$\|T\| = \|R(T^{-1}, 0)\| = 1/d(0, \sigma(T^{-1})) = R_{sp}(T).$$

Contradiction. Hence T^{-1} is not paranormal.

II. Topological properties of $\mathcal{P}$. Recall that $\mathcal{N}$ is the set of all normal operators on H, $\mathcal{P}$ is the set of all paranormal operators on H, $\mathcal{L}$ is the set of all $T \in B(H)$ such that $\operatorname{co} \sigma(T) = W(T)$, and $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L}$. It will always be assumed that $B(H)$ has the uniform operator (norm) topology.

The following notation will be used in this section: If S is a compact subset of the complex numbers $\mathbf{C}$ and if $\epsilon > 0$, then let $S + (\epsilon) = \{z: d(z, S) < \epsilon\}$. If S and S_n, $n = 1, 2, 3, \cdots$, are compact sets in $\mathbf{C}$, then the sequence $\{S_n\}$ approaches S, written $S_n \longrightarrow S$, if for every $\epsilon > 0$ there exists a positive integer N such that, for $n \geq N$, $S_n \subseteq S + (\epsilon)$ and $S \subseteq S_n + (\epsilon)$.

In general $\sigma(T)$ is not a continuous function of T in $B(H)$ (see [7, problem 85]), but $\sigma(T)$ is continuous if we restrict T to $\mathcal{P}$.

Theorem 2.1. *If $\{T_n\}$ is a sequence of paranormal operators approaching the operator T in norm, then $\sigma(T_n) \longrightarrow \sigma(T)$ as $n \longrightarrow \infty$.*

To prove this theorem we need the following lemma from [7, problem 86].

Lemma. *If $T \in B(H)$ and $\epsilon > 0$, then there exists $\delta > 0$ such that if $S \in B(H)$ and $\|T - S\| < \delta$, then $\sigma(S) \subseteq \sigma(T) + (\epsilon)$.*

Proof of Theorem 2.1. We know by the lemma that for each $\epsilon > 0$ there exists a positive integer N such that, for $n \geq N$, $\sigma(T_n) \subseteq \sigma(T) + (\epsilon)$. Therefore, to show $\sigma(T_n) \to \sigma(T)$, it suffices to show that for each $\epsilon > 0$ there exists a positive integer N such that $\sigma(T) \subseteq \sigma(T_n) + (\epsilon)$ for all $n \geq N$. If this does not hold, then without loss of generality we may assume that there exists $\epsilon > 0$ and a sequence $\{z_n\} \subseteq \sigma(T)$ such that $d(z_n, \sigma(T_n)) \geq \epsilon$ for all n. Since $\sigma(T)$ is compact, we may assume $z_n \longrightarrow z \in \sigma(T)$. If $|z_n - z| < \epsilon/2$, then

$$d(z, \sigma(T_n)) \geq d(z_n, \sigma(T_n)) - |z - z_n| \geq \epsilon - \epsilon/2 = \epsilon/2.$$

Hence

$$\|R(T_n, z)\| = 1/d(z, \sigma(T_n)) \leq 2/\epsilon.$$

Now choose m so that $\|(T_m - T)R(T_m, z)\| < 1$, then $I - (T_m - T)R(T_m, z)$ is invertible [7, problem 173]. Let

$$A = R(T_m, z)(I - (T_m - T)R(T_m, z))^{-1}.$$

Then $A(T - zI) = (T - zI)A = I$ so that $z \in \rho(T)$. Contradiction.

Remark. Note that the full strength of the paranormality assumption was not used in this proof, and that one can easily devise various larger classes of operators on which $\sigma(T)$ is continuous.

Theorem 2.2. $\mathcal{P}$ *is an arcwise connected, closed subset of* $B(H)$.

Proof. Since $T \in \mathcal{P}$ implies $aT \in \mathcal{P}$ for every complex number a, we see that the ray in $B(H)$ through T is contained in $\mathcal{P}$. Therefore $\mathcal{P}$ is arcwise connected.

Suppose $T_n \to T$, $\{T_n\}$ a sequence of operators in $\mathcal{P}$, and $T \in B(H)$. Let $z \in \rho(T)$. By the lemma to Theorem 2.1,

$$\limsup_{n\to\infty} \frac{1}{d(z, \sigma(T_n))} \le \frac{1}{d(z, \sigma(T))}.$$

Therefore, since $\|R(T_n, z)\| = 1/d(z, \sigma(T_n))$ whenever $z \in \rho(T_n)$, there exists a positive integer N such that the sequence $\{\|R(T_n, z)\|: n \ge N\}$ is bounded. Then, since $R(T, z) - R(T_n, z) = R(T, z)(T - T_n)R(T_n, z)$, $\|R(T_n, z)\| \to \|R(T, z)\|$ as $n \to \infty$. Consequently,

$$\|R(T, z)\| = \lim_{n\to\infty} \|R(T_n, z)\| = \lim_{n\to\infty} \frac{1}{d(z, \sigma(T_n))} \le \frac{1}{d(z, \sigma(T))}.$$

Since the general $\|R(T, z)\| \ge 1/d(z, \sigma(T))$, T is paranormal.

Theorem 2.3. $\mathcal{L}$ *is an arcwise connected, closed subset of* $B(H)$.

Proof. Since $T \in \mathcal{L}$ implies that $aT \in \mathcal{L}$ for every complex number a, $\mathcal{L}$ is arcwise connected.

Let $T_n \to T$, $\{T_n\} \subseteq \mathcal{L}$ and $T \in B(H)$. Since $|(T_n x, x) - (Tx, x)| \le \|T_n - T\|$ for $\|x\| = 1$, $W(T_n) \subseteq W(T) + (2\,\|T - T_n\|)$ and $W(T) \subseteq W(T_n) + (2\,\|T - T_n\|)$. Consequently, $W(T_n) \to W(T)$. Let $\epsilon > 0$, then by the lemma to Theorem 2.1 there exists a positive integer N such that $\sigma(T_n) \subseteq \sigma(T) + (\epsilon)$ for all $n \ge N$. Therefore, for $n \ge N$, $\operatorname{co} \sigma(T_n) \subseteq \operatorname{co} \sigma(T) + (\epsilon)$ and hence

$$W(T) = \lim_{n\to\infty} W(T_n) = \lim_{n\to\infty} \operatorname{co} \sigma(T_n) \subseteq \operatorname{co} \sigma(T) + (\epsilon).$$

Since $\epsilon > 0$ is arbitrary, $W(T) \subseteq \operatorname{co} \sigma(T)$. Since in general $\operatorname{co} \sigma(T) \subseteq W(T)$, $T \in \mathcal{L}$.

Let $\mathcal{N}$ be the set of all normal operators on H. Since $\|T_n - T\| \to 0$ implies $\|T_n^* - T^*\| \to 0$, $\mathcal{N}$ is closed in the uniform operator topology on $B(H)$. Since $T \in \mathcal{N}$ implies $aT \in \mathcal{N}$ for any complex a, $\mathcal{N}$ is arcwise connected.

We already know that $\mathcal{N} \subseteq \mathcal{P} \subseteq \mathcal{L} \subseteq B(H)$. We will now investigate the relative topological properties of these four sets. Stampfli [20, Theorem C] has shown that $\mathcal{N} = \mathcal{P}$ when $\dim H < \infty$. However, when $\dim H = \infty$, then $\mathcal{N}$ is a very "thin" subset of $\mathcal{P}$.

Theorem 2.4. $\mathcal{N}$ *is a nowhere dense subset of* $\mathcal{P}$ *when* $\dim H = \infty$.

Proof. Since $\mathcal{N}$ is closed, to show that $\mathcal{N}$ is a nowhere dense subset of $\mathcal{P}$, it suffices to show that $\mathcal{N}$ has empty interior in $\mathcal{P}$. Let $T \in \mathcal{N}$ and let $\epsilon > 0$.

First suppose that T has an eigenvalue of infinite multiplicity. We may assume that the eigenvalue is zero. Let M be the eigenspace of zero. Then $\dim M = \infty$, M reduces T, and we can write $T = B \oplus Z$ where Z is the zero operator on M. Let $S = \left[\begin{smallmatrix} 0 & \epsilon \\ 0 & 0 \end{smallmatrix}\right] \oplus N$ be a nonnormal paranormal operator [see Theorem 1.2] on M with N a normal operator such that $\sigma(N) = W\left[\begin{smallmatrix} 0 & \epsilon \\ 0 & 0 \end{smallmatrix}\right]$. Then $B \oplus S$ is a nonnormal paranormal operator such that $\|T - B \oplus S\| = \|B \oplus Z - B \oplus S\| = \|S\| = \epsilon$. The last equality holds since $\|N\| = R_{sp}(N) = \epsilon/2$ and $\left\|\left[\begin{smallmatrix} 0 & \epsilon \\ 0 & 0 \end{smallmatrix}\right]\right\| = \epsilon$. Therefore, since $\epsilon > 0$ is arbitrary, T is not contained in the interior of $\mathcal{N}$ in $\mathcal{P}$.

If $\sigma(T)$ is finite and $T \in \mathcal{N}$, then $\sigma(T) = \sigma_p(T)$ and T has an eigenvalue of infinite multiplicity. We therefore assume that $\sigma(T)$ is infinite and that zero is an accumulation point of $\sigma(T)$. Let D be the open disc about zero of radius $\epsilon/2$. Let E be the resolution of the identity for T so that $T = \int_{\sigma(T)} z\, dE_z$. Let $M = (E(D))(H)$, $P = \sigma(T) - D$, and let $A = \int_P z\, dE_z$. Then M reduces T, $\dim M = \infty$, and A is a normal operator. Let Z be the zero operator on M. Then $A \oplus Z$ is a normal operator with zero an eigenvalue of infinite multiplicity, and

$$\|T - A \oplus Z\| = \left\| \int_D z\, dE_z \right\| \leq \epsilon/2.$$

By the first part of this proof, there exists a nonnormal paranormal operator S such that $\|A \oplus Z - S\| < \epsilon/2$. Then

$$\|T - S\| \leq \|T - A \oplus Z\| + \|A \oplus Z - S\| < \epsilon.$$

Therefore, since $\epsilon > 0$ is arbitrary, T is not contained in the interior of $\mathcal{N}$ in $\mathcal{P}$. Hence the interior of $\mathcal{N}$ in $\mathcal{P}$ is empty.

Define C_2 to be the set of all operators $T \in \mathcal{L}$ with $W(T)$ a closed line segment or a point. For $k = 3, 4, 5, \cdots$, let C_k be the set of all operators $T \in \mathcal{L}$ such that $W(T)$ is the convex hull of a nondegenerate polygon with k sides. If $T \in C_k$, then each vertex of $W(T)$ must be in the spectrum of T [1]. S. Hildebrandt [10, Theorem 2] has shown that, if $z \in \sigma_p(T) \cap \partial W(T)$, then z is a normal eigenvalue of T. Thus for $T \in C_k$, the vertices of $W(T)$ are normal eigenvalues of T, when $\dim H < \infty$. Hence, all the operators in $C_n \cup C_{n-1}$ are normal operators when $\dim H = n < \infty$.

In [10, Theorem 9] S. Hildebrandt showed that $\mathcal{N} = \mathcal{P} = \mathcal{L}$ when $\dim H \leq 4$, and that $\mathcal{N} \neq \mathcal{L}$ for $5 \leq \dim H < \infty$. The following theorem gives more information on how $\mathcal{P}$ and $\mathcal{L}$ are related when $5 \leq \dim H < \infty$. Recall that $\mathcal{N} = \mathcal{P}$ for $\dim H < \infty$.

Theorem 2.5. *If $5 \leq \dim H = n < \infty$, then the interior of $\mathcal{P}$ in $\mathcal{L}$ equals $C_n \cup C_{n-1}$.*

Proof. Suppose $T \in C_n \cup C_{n-1}$. Since $C_n \cup C_{n-1} \subseteq \mathcal{N}$, T is normal. First, we show that there exists $\epsilon > 0$ such that whenever $S \in \mathcal{L}$ and $\|T - S\| < \epsilon$, then

$S \in C_n \cup C_{n-1}$. To show this, suppose the statement were false. Then there would exist $\{S_n\} \subseteq \mathfrak{L}$ such that $\|T - S_n\| \to 0$ and $S_n \in \bigcup_{i=2}^{n-2} C_i$. Then, since $W(S_n) \to W(T)$, $T \in \bigcup_{i=2}^{n-2} C_i$. Contradiction. Hence, T is an interior point of $\mathfrak{P}$ in $\mathfrak{L}$.

Let T be contained in the interior of $\mathfrak{P} = \mathfrak{N}$ in $\mathfrak{L}$. Suppose $T \notin C_n \cup C_{n-1}$. Let $\epsilon > 0$. Since $\operatorname{co}\sigma(T) = W(T)$, $T \in C_k$, for some $k \leq n-2$. Since $\dim H \geq 5$ and since T is a normal operator in C_k, there exists a normal operator N such that

1. $\|T - N\| < \epsilon/2$,
2. $W(N)$ is a polygon with at least three sides, and
3. N has at least two eigenvalues z, w contained in the interior of $W(N)$.

Write $N = A \oplus B$ where B can be written as $B = \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix}$. Let $a > 0$ and let $C = \begin{bmatrix} z & a \\ 0 & w \end{bmatrix}$; then $\|B - C\| = \left\|\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right\| = a$. Choose $a > 0$ small enough so that $W(C) \subseteq W(N)$ and so that $a < \epsilon/2$. Then since $W(A) = W(N)$ and $\sigma(A \oplus C) = \sigma(N)$, $\operatorname{co}\sigma(A \oplus C) = W(A \oplus C)$. Hence $A \oplus C \in \mathfrak{L}$. Since $A \oplus C$ is not normal, and since

$$\|T - A \oplus C\| \leq \|T - N\| + \|N - A \oplus C\| \leq \epsilon/2 + \epsilon/2 = \epsilon,$$

T is not an interior point of $\mathfrak{P}$ in $\mathfrak{L}$. Contradiction. Hence $T \in C_n \cup C_{n-1}$.

It is an open question as to what the interior of $\mathfrak{P}$ in $\mathfrak{L}$ is when $\dim H = \infty$. However, it can be shown that $\mathfrak{P} \neq \mathfrak{L}$ when $\dim H = \infty$.

Theorem 2.6. $\mathfrak{P} \neq \mathfrak{L}$ *when* $\dim H = \infty$.

Proof. Write $H = M \oplus M^{\perp}$ where $\dim M = 5$. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

where a, b, c are three complex numbers that form a triangle with $W(A)$ contained in the interior of the triangle. Consider $A \oplus N$ as an operator on M and observe that $\operatorname{co}\sigma(A \oplus N) = W(N) = W(A \oplus N)$. Since $A \oplus N$ is not normal and since $\dim M < \infty$, $A \oplus N$ is not paranormal. Hence there exists $z \in \rho(A \oplus N)$ such that

$$\|R(A \oplus N, z)\| > 1/d(z, \sigma(A \oplus N)).$$

Let I be the identity operator on $M^{\perp}$ and let $T = A \oplus N \oplus aI$. Then, $\sigma(T) = \sigma(A \oplus N)$ and $W(T) = W(A \oplus N)$. Therefore $T \in \mathfrak{L}$. Since $d(z, \sigma(T)) \leq |z - a|$,

$$\begin{aligned} \|R(T, z)\| &= \operatorname{Max}\{\|R(A \oplus N, z)\|, |z - a|^{-1}\} \\ &= \|R(A \oplus N, z)\| > 1/d(z, \sigma(T)). \end{aligned}$$

Therefore T is not paranormal.

We now show that if $\dim H \geq 2$, then $\mathfrak{L}$ is a nowhere dense ("thin") subset of $B(H)$. Once this is shown, it follows immediately that $\mathfrak{N}$ and $\mathfrak{P}$ are also nowhere dense subsets of $B(H)$.

Theorem 2.7. $\mathfrak{L}$ *is a nowhere dense subset of* $B(H)$ *when* $\dim H \geq 2$.

To prove this theorem we need the following two technical lemmas.

Lemma 1. *If z_1 and z_2 are distinct, normal approximate eigenvalues of $T \in B(H)$, then there exist sequences $\{x_n\}$ and $\{y_n\}$ of unit vectors in H such that*

1. $(x_n, y_n) = 0$ *for all* n,
2. $\|(T - z_1 I)x_n\| \to 0$ *as* $n \to \infty$, *and*
3. $\|(T - z_2 I)y_n\| \to 0$ *as* $n \to \infty$.

Lemma 2. *If $T \in \mathfrak{L}$ such that there exists distinct $a, b \in \partial W(T) \cap \sigma_p(T)$, then T is not contained in the interior of $\mathfrak{L}$.*

Proof of Theorem 2.7. Since $\mathfrak{L}$ is closed (Theorem 2.3), to show that $\mathfrak{L}$ is nowhere dense it suffices to show that $\mathfrak{L}$ has empty interior.

We first show that if T is in the interior of $\mathfrak{L}$, then $\sigma(T)$ must contain at least two points. Suppose $T \in \mathfrak{L}$ and $\sigma(T) = \{a\}$. Then $((T - aI)x, x) = 0$ for all $x \in H$ so that $T = aI$. Since $\dim H \geq 2$, write $H = M \oplus M^{\perp}$ where $\dim M = 2$. Let $b > 0$ and define $A \in B(H)$ as

$$A = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \text{ on } M, \qquad \text{and } A = 0 \quad \text{on } M^{\perp}.$$

Then $\sigma(T + A) = \{a\}$ and, since $b \neq 0$, $\{a\} \neq W(T + A)$. Therefore $T + A \notin \mathfrak{L}$. Since $\|A\| = b > 0$ is arbitrary, $T \notin$ interior $\mathfrak{L}$.

With the above remark completed, we can now finish the proof of Theorem 2.7. Suppose the theorem were false and there exists $T \in$ interior $\mathfrak{L}$. Then there exists $\epsilon > 0$ such that whenever $V \in B(H)$ and $\|T - V\| < \epsilon$, then $V \in \mathfrak{L}$. From the above remark $\sigma(T)$ must contain at least two points. There must be at least two extreme points of $W(T)$, since extreme points of $W(T)$ for $T \in \mathfrak{L}$ are extreme points of $\sigma(T)$. Hence, after a rotation, if necessary, we may assume there exists z_1, $z_2 \in \sigma_\pi(T) \cap \partial W(T)$ such that

1. $\operatorname{Re} z_1 = \inf \operatorname{Re} W(T)$,
2. $\operatorname{Re} z_2 = \sup \operatorname{Re} W(T)$, and
3. $\operatorname{Re} z_1 < \operatorname{Re} z_2$.

Since $z_1, z_2 \in \partial W(T) \cap \sigma_\pi(T)$, z_1 and z_2 are normal approximate eigenvalues of T [10, Theorem 2, p. 233]. By Lemma 1 there exist unit vectors $x, y \in H$ such that $(x, y) = 0$, $\|(T - z_1 I)x\| < \epsilon/8$, and $\|(T - z_2 I)y\| < \epsilon/8$. Let M be the closed subspace spanned by $\{x, y\}$. Define $C \in B(H)$ as

$Cx = -(\epsilon/4)x$,

$Cy = +(\epsilon/4)y$, and

$Cz = 0$ for all $z \in M^{\perp}$.

Since $\|C\| \leq \epsilon/2$, $T + C \in \mathfrak{L}$. Since

$$((T + C)x, x) = (Tx, x) - \epsilon/4 \quad \text{and}$$
$$|(Tx, x) - z_1| \leq \|(T - z_1 I)x\| < \epsilon/8,$$

we obtain $\inf \operatorname{Re} W(T + C) < \operatorname{Re} z_1$. Since $T + C \in \mathfrak{L}$, there exists $a \in \sigma_\pi(T + C) \cap \partial W(T + C)$ such that $\operatorname{Re} a = \inf \operatorname{Re} W(T + C)$. Since C is a compact operator, Weyl's spectral inclusion theorem [7, problem 143] yields $\sigma(T + C) - \sigma_p(T + C) \subseteq \sigma(T)$. Therefore, $a \in \sigma_p(T + C)$ so that $a \in \sigma_p(T + C) \cap \partial W(T + C)$. Similarly one shows there exists

$$b \in \sigma_p(T + C) \cap \partial W(T + C)$$

such that $\operatorname{Re} b = \sup W(T + C) > \operatorname{Re} z_2$, and hence $a \neq b$. By Lemma 2, there exists $S \in B(H)$ such that $\|S\| < \epsilon/2$ and $T + C + S \notin \mathfrak{L}$. However, $\|T - (T + C + S)\| \leq \|C\| + \|S\| < \epsilon$ and so by assumption $T + C + S \in \mathfrak{L}$. Contradiction.

Proof of Lemma 1. There exist sequences $\{w_n\}$ and $\{y_n\}$ of unit vectors in H such that $\|(T - z_1 I)w_n\| \longrightarrow 0$ and $\|(T - z_2 I)y_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Then

$$|(z_1 - z_2)(w_n, y_n)| = |(z_1 w_n, y_n) - (w_n, z_2^* y_n)|$$
$$\leq |((T - z_1 I)w_n, y_n)| + |(w_n, (T - z_2 I)^* y_n)| \leq \|(T - z_1 I)w_n\| + \|(T - z_2 I)^* y_n\|.$$

Therefore, $|(z_1 - z_2)(w_n, y_n)| \longrightarrow 0$ as $n \longrightarrow \infty$. Since $z_1 \neq z_2$, $(w_n, y_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

There exist complex numbers a_n and b_n and unit vectors x_n in H such that $w_n = a_n y_n + b_n x_n$, $|a_n|^2 + |b_n|^2 = 1$, and $(x_n, y_n) = 0$. From the above paragraph we have that $a_n \longrightarrow 0$, so $|b_n| \longrightarrow 1$. Therefore, $\|(T - z_1 I)x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof of Lemma 2. Let $\epsilon > 0$. Since $a, b \in \sigma_p(T) \cap \partial W(T)$, a and b are normal eigenvalues of T [10, Theorem 2, p. 233]. Let $u, v \in H$ be unit vectors such that $Tu = au$ and $Tv = bv$. Then $(u, v) = 0$ and the closed subspace N spanned by $\{u, v\}$ reduces T. Define $S \in B(H)$ as

$$Su = \epsilon v, \quad Sv = 0, \quad Sz = 0 \quad \text{for all } z \in N^\perp.$$

Then we may write $T + S = A \oplus B$ corresponding to $H = N \oplus N^\perp$. Then the matrix representation of A relative to $\{u, v\}$ is $A = \left[\begin{smallmatrix} a & \epsilon \\ 0 & b \end{smallmatrix}\right]$. Hence $\sigma(A) = \{a, b\} \subseteq \sigma(T)$. Clearly $\sigma(B) \subseteq \sigma(T)$. Therefore,

$$\operatorname{co} \sigma(T + S) \subseteq \operatorname{co} \sigma(T) = W(T).$$

Since A is not a normal operator, $W(A)$ is the convex hull of a nondegenerate ellipse (i.e., not a straight line) with foci at a and b (see [1]). Since $W(A) \subseteq W(T + S)$, we must have $\operatorname{co} \sigma(T + S) \neq W(T + S)$. Therefore, $T + S \notin \mathfrak{L}$. Thus, since $\|S\| = \epsilon > 0$ is arbitrary, $T \notin$ interior $\mathfrak{L}$.

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DEPARTMENT OF MATHEMATICS, IOWA STATE UNIVERSITY, AMES, IOWA 50010